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COMMENT

A renormalisation group approach to the Potts model on the Koch curve

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Abstract. A three-parameter decimation renormalisation is constructed and used to produce exact RG recursion relations for the Potts model on the Koch curve including all the nearest-neighbour interactions. Only two trivial fixed points are obtained. We show that no phase transition occurs at finite temperature.

Recently, a great deal of attention has been paid to critical behaviour on fractals. The reason is that the percolation cluster at threshold may be represented by a random fractal. More generally, a fractal can be considered as a crude approximation for a disordered system and results obtained for fractals may have some relevance for real random systems.

A regular fractal is much more easy to treat than a random one. Many research works focus on the study of critical properties of spin models on regular fractals. Some very powerful approaches, such as transfer matrix techniques (Andrade and Salinas 1984), decimation renormalisation (Gefen *et al* 1980, 1983, 1984a, b) and a Migdal-Kadanoff bond-moving method (Gefen *et al* 1984a) were used. Gefen *et al* have treated the Ising model on a Koch curve with a decimation procedure (Gefen *et al* 1980). Unfortunately, they do not include all nearest-neighbour interactions. Andrade and Salinas allow all nearest-neighbour interactions not only between, for example, ⟨5, 6⟩, ⟨6, 7⟩, ⟨6, 8⟩, ⟨7, 8⟩ and ⟨8, 9⟩, but also between ⟨8, 10⟩ (see figure 2). They employ a transfer matrix approach to calculate the partition function and obtain an exact analytic result in zero magnetic field. However, this method in this case will lead to many more complications and will fail if a single-parameter decimation procedure is used.

In this comment we report a three-parameter renormalisation group treatment for the q-state Potts model (the Ising model is a special case of $q=2$) on the Koch curve. The first two stages of the structure of the Koch curve are shown in figures 1 and 2. In our model all nearest-neighbour interactions will be considered. To obtain exact recursion relations we introduce three kinds of coupling parameters J, K and L: (a) K, for nearest neighbours along the curve, which correspond to bonds forming triangles,

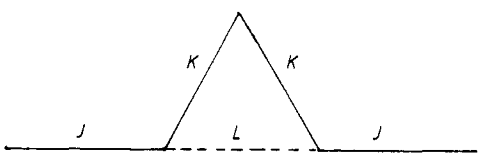

Figure 1. Koch curve for the first stage of the construction.

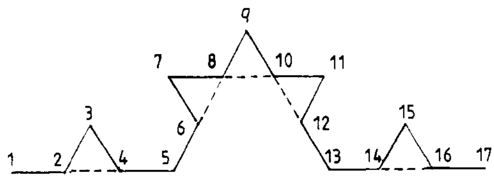

Figure 2. Koch curve for the second stage of the construction.

such as ⟨2, 3⟩, ⟨3, 4⟩, ⟨6, 7⟩, ⟨8, 9⟩, etc; (*b*) *J*, for nearest neighbours along the curve, but which do not correspond to bonds forming triangles, such as ⟨1, 2⟩, ⟨4, 5⟩, ⟨5, 6⟩, etc, and (*c*) *L*, for nearest neighbours shown in broken lines in the figures, which do not correspond to bonds along the curve, such as ⟨2, 4⟩, ⟨6, 8⟩, ⟨8, 10⟩, etc (see figure 2). We write the q-state Potts model Hamiltonian on the Koch curve as follows:

$$
\begin{aligned}
-\frac{H}{kT}=J\Bigg(&\sum_{l=1}^{N-2}\sum_{i=1}^{2^{2l-1}}\left(\delta_{a_r{}^{N-l}{}_{(2i-1)},a_4{}^{N-l}{}_{(2i-1)+1}}+\delta_{a_4{}^{N-l}{}_{(2i-1)+1},a_4{}^{N-l}{}_{(2i-1)+2}}\right)\\
&+\sum_{i=1}^{2^{2N-3}}\left(\delta_{a_{4(2i-1)},a_{4(2i-1)+1}}+\delta_{a_{4(2i-1)+1},a_{4(2i-1)+2}}\right)+\delta_{a_1,a_2}+\delta_{a_{4^N},a_{4^N+1}}\Bigg)\\
&+K\sum_{l=1}^{N}\sum_{i=1}^{4^{l-1}}\left(\delta_{a_{4^{N-l}(4i-2)},a_{4^{N-l}(4i-2)+1}}+\delta_{a_{4^{N-l}(4i-2)+1},a_{4^{N-l}(4i-2)+2}}\right)\\
&+L\sum_{l=1}^{N}\sum_{i=1}^{4^{l-1}}\delta_{a_{4^{N-l}(4i-2)},a_{4^{N-l}(4i-2)+2}}
\end{aligned}
\tag{1}
$$

where N indicates the Nth step of the construction.

Before proceeding with the decimation procedure, we first analyse the construction of the Koch curve. It will be found that the two units shown in figure 3 are the basic components which compose any construction stage of the Koch curve.

Let us define four matrices as follows:

$$
\begin{aligned}
\langle a|\hat{A}|a'\rangle &= \exp(L\delta_{a,a'})\\
\langle a|\hat{B}|a'\rangle &= \exp(K\delta_{a,a'})\\
\langle a|\hat{C}|a'\rangle &= \exp(J\delta_{a,a'})\\
\langle a|\hat{D}|a'\rangle &= \langle a|\hat{B}^2|a'\rangle\langle a|\hat{A}|a'\rangle.
\end{aligned}
\tag{2}
$$

Obviously, all these matrices are symmetric about their diagonal and commute with one another, and the product of these matrices still keeps the above properties. By means of (2), we get the following expressions which we will use later:

$$[\hat{C}^2\hat{D}]_{\alpha\alpha}=\mathrm{e}^L(\mathrm{e}^{2K}+q-1)(\mathrm{e}^{2J}+q-1)+(q-1)(2\mathrm{e}^J+q-2)(2\mathrm{e}^K+q-2)$$

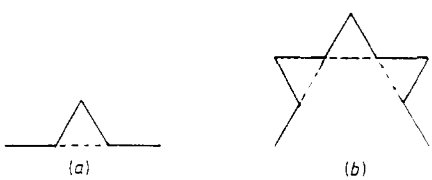

Figure 3. The two basic units which compose any construction stage of the Koch curve. (*a*) Unit 1, (*b*) unit 2.

$$
\begin{aligned}
[\hat{C}^2\hat{D}]_{\alpha\beta} &= (e^{2J}+q-1)(2e^K+q-2)+e^L(e^{2K}+q-1)(2e^J+q-2)\\
&\quad +(q-2)(2e^J+q-2)(2e^K+q-2)\\
[\hat{C}\hat{B}\hat{D}]_{\alpha\alpha} &= e^L(e^{2K}+q-1)(e^{J+K}+q-1)+(q-1)(2e^{2K}+q-2)(e^J+e^K+q-2)\\
[\hat{C}\hat{B}\hat{D}]_{\alpha\beta} &= e^L(e^{2K}+q-1)(e^J+e^K+q-2)\\
&\quad +(2e^K+q-2)[(e^J+q-2)(e^K+q-2)+q-1] \qquad (3)\\
[\hat{C}\hat{D}]_{\alpha\alpha} &= e^{J+L}(e^{2K}+q-1)+(q-1)(2e^K+q-2)\\
[\hat{C}\hat{D}]_{\alpha\beta} &= e^L(e^{2K}+q-1)+(e^J+q-2)(2e^K+q-2)\\
[\hat{B}]_{\alpha\alpha} &= e^K\\
[\hat{B}]_{\alpha\beta} &= 1
\end{aligned}
$$

where α and β indicate the state of Potts spin.

We now proceed to present the decimation renormalisation. To construct recursion relations we consider the two basic units as the components of the original nth construction stage of the Koch curve. By integrating over the spins on some sites and leaving the remainder fixed, we obtain a renormalised construction which has the components of the $(n-1)$th construction stage. The RG procedure is shown schematically in figures 4(a) and (b). They respectively correspond to the following expressions:

$$
\begin{aligned}
F\exp(J'\delta_{a_1,a_5}) &= \sum_{a_2a_3a_4}\exp[J(\delta_{a_1,a_2}+\delta_{a_4,a_5})+K(\delta_{a_2,a_3}+\delta_{a_3,a_4})+L(\delta_{a_2,a_4})]\\
&= \langle a_1|\hat{C}^2\hat{D}|a_5\rangle \qquad (4)
\end{aligned}
$$

$$
\begin{aligned}
F'\exp[K'(\delta_{a_1,a_5}+\delta_{a_5,a_9})+L'\delta_{a_1,a_9}] &= \sum_{\substack{a_2a_3a_4\\a_6a_7a_8}}\exp[L(\delta_{a_2,a_4}+\delta_{a_4,a_6}+\delta_{a_6,a_8})\\
&\quad +K(\delta_{a_2,a_3}+\delta_{a_3,a_4}+\delta_{a_4,a_5}+\delta_{a_5,a_6}+\delta_{a_6,a_7}+\delta_{a_7,a_8})+J(\delta_{a_1,a_2}+\delta_{a_8,a_9})]\\
&= (e^L-1)\sum_{a_6}\langle a_1|\hat{C}\hat{D}|a_6\rangle\langle a_6|\hat{C}\hat{D}|a_9\rangle(\langle a_6|\hat{B}|a_5\rangle)^2+\langle a_1|\hat{C}\hat{B}\hat{D}|a_5\rangle\langle a_5|\hat{C}\hat{B}\hat{D}|a_9\rangle \qquad (5)
\end{aligned}
$$

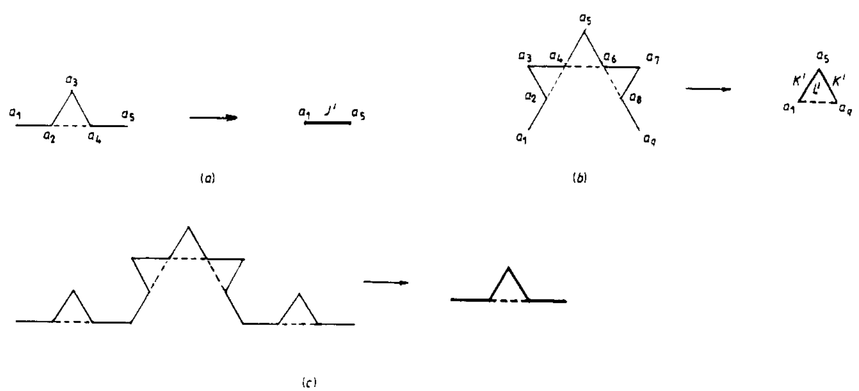

Figure 4. A schematic RG transformation. (a) Decimating a_2, a_3 and a_4 and leaving a_1 and a_5 fixed in unit 1; (b) decimating a_2, a_3, a_4, a_6, a_7 and a_8 and leaving a_1, a_5 and a_q fixed in unit 2; (c) the RG transformation from the second to the first construction stage.

and give rise to the recursion relations as follows:

$$
\begin{aligned}
\mathrm{e}^{-J'} &= P_2(J, K, L)/P_1(J, K, L) \\
\mathrm{e}^{-2K'} &= M_2(J, K, L)/M_1(J, K, L) \\
\mathrm{e}^{-L'-K'} &= M_3(J, K, L)/M_1(J, K, L)
\end{aligned} \tag{6}
$$

where P_1, P_2, M_1, M_2 and M_3 represent respectively

$$P_1 = \langle a_1|\hat{C}^2\hat{D}|a_5\rangle|_{a_1=a_5} = [\hat{C}^2\hat{D}]_{\alpha\alpha}$$

$$P_2 = \langle a_1|\hat{C}^2\hat{D}|a_5\rangle|_{a_1\neq a_5} = [\hat{C}^2\hat{D}]_{\alpha\beta}$$

$$
\begin{aligned}
M_1 &= (\langle a_1|\hat{C}\hat{B}\hat{D}|a_5\rangle\langle a_5|\hat{C}\hat{B}\hat{D}|a_9\rangle + (\mathrm{e}^L - 1)\sum_{a_6}\langle a_1|\hat{C}\hat{D}|a_6\rangle\langle a_6|\hat{C}\hat{D}|a_9\rangle(\langle a_6|\hat{B}|a_5\rangle)^2)|_{a_1=a_5=a_q} \\
&= (\mathrm{e}^L-1)[([\hat{C}\hat{D}]_{\alpha\alpha})^2([\hat{B}]_{\alpha\alpha})^2 + (q-1)([\hat{C}\hat{D}]_{\alpha\beta})^2([\hat{B}]_{\alpha\beta})^2] + ([\hat{C}\hat{B}\hat{D}]_{\alpha\alpha})^2
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
M_2 &= (\langle a_1|\hat{C}\hat{B}\hat{D}|a_5\rangle\langle a_5|\hat{C}\hat{B}\hat{D}|a_9\rangle + (\mathrm{e}^L - 1)\sum_{a_6}\langle a_1|\hat{C}\hat{D}|a_6\rangle\langle a_6|\hat{C}\hat{D}|a_9\rangle(\langle a_6|\hat{B}|a_5\rangle)^2)|_{a_1=a_q\neq a_5} \\
&= (\mathrm{e}^L-1)\{([\hat{C}\hat{D}]_{\alpha\alpha})^2([\hat{B}]_{\alpha\beta})^2 + ([\hat{C}\hat{D}]_{\alpha\beta})^2[([\hat{B}]_{\alpha\alpha})^2 \\
&\quad + (q-2)([\hat{B}]_{\alpha\beta})^2]\} + ([\hat{C}\hat{B}\hat{D}]_{\alpha\beta})^2
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
M_3 &= (\langle a_1|\hat{C}\hat{B}\hat{D}|a_5\rangle\langle a_5|\hat{C}\hat{B}\hat{D}|a_9\rangle + (\mathrm{e}^L - 1)\sum_{a_6}\langle a_1|\hat{C}\hat{D}|a_6\rangle\langle a_6|\hat{C}\hat{D}|a_9\rangle(\langle a_6|\hat{B}|a_5\rangle)^2)|_{a_1=a_5\neq a_q} \\
&= (\mathrm{e}^L-1)\{[\hat{C}\hat{D}]_{\alpha\alpha}[\hat{C}\hat{D}]_{\alpha\beta}[([\hat{B}]_{\alpha\alpha})^2 + ([\hat{B}]_{\alpha\beta})^2] \\
&\quad + (q-2)([\hat{C}\hat{D}]_{\alpha\beta})^2([\hat{B}]_{\alpha\beta})^2\} + [\hat{C}\hat{B}\hat{D}]_{\alpha\alpha}[\hat{C}\hat{B}\hat{D}]_{\alpha\beta}.
\end{aligned} \tag{9}
$$

In order to study the critical behaviour we must find the fixed points of equation (6). It is clear that $J^* = K^* = L^* = \infty$ is one of the fixed points because the total order of powers of e^X in P_1 is higher than one in P_2 and the total order of powers of e^X in M_1 is higher than one in M_2 and M_3, where e^X denotes e^{L^*}, e^{K^*} and e^{J^*}. This fixed point corresponds to a completely ordered phase of the system at zero temperature. To find the other fixed point we rewrite the expressions in (6) as

$$
\begin{aligned}
&(\mathrm{e}^{J^*}-1)(\mathrm{e}^{J^*}+q-1)[\mathrm{e}^{L^*}(2\mathrm{e}^{K^*}+q-2)+(\mathrm{e}^{2K^*}+q-1)(\mathrm{e}^{J^*}+q-2)]=0 \\
&(\mathrm{e}^{K^*}-1)[\mathrm{e}^{L^*}(\mathrm{e}^{2K^*}+q-1)+(\mathrm{e}^{J^*}+q-2)(2\,\mathrm{e}^{K^*}+q-2)] \\
&\qquad\times\{[\mathrm{e}^{L^*}(\mathrm{e}^{2K^*}+q-1)+(\mathrm{e}^{J^*}+q-2)(2\,\mathrm{e}^{K^*}+q-2)] \\
&\qquad\times[(\mathrm{e}^{2K^*}+q-1)(\mathrm{e}^{L^*+K^*}+\mathrm{e}^{L^*}+q-2)+(q-2)\mathrm{e}^{K^*}(\mathrm{e}^{K^*}+q-1)] \\
&\qquad+2\,\mathrm{e}^{K^*}[\mathrm{e}^{J^*+L^*}(\mathrm{e}^{2K^*}+q-1)+(q-1)(2\,\mathrm{e}^{K^*}+q-2)](\mathrm{e}^{K^*}+q-1)\}=0 \\
&\mathrm{e}^{K^*}[\mathrm{e}^{J^*+L^*}(\mathrm{e}^{2K^*}+q-1)+(q-1)(2\,\mathrm{e}^{K^*}+q-2)] \\
&\qquad\times[\mathrm{e}^{L^*}(\mathrm{e}^{2K^*}+q-1)+(\mathrm{e}^{J^*}+q-2)(2\mathrm{e}^{K^*}+q-2)] \\
&\qquad\times\{\mathrm{e}^{L^*}[\mathrm{e}^{L^*+2K^*}+\mathrm{e}^{L^*}+(q-2)(\mathrm{e}^{K^*}+1)]-2(q-1)\} \\
&\qquad+[\mathrm{e}^{L^*}(\mathrm{e}^{2K^*}+q-1)+(\mathrm{e}^{J^*}+q-2)(2\,\mathrm{e}^{k^*}+q-2)]^2 \\
&\qquad\times\{\mathrm{e}^{L^*+K^*}[(\mathrm{e}^{L^*}-1)(q-2)+(q-1)(\mathrm{e}^{K^*}+q-2)]-(q-1)(\mathrm{e}^{L^*}+q-2)\} \\
&\quad=0.
\end{aligned} \tag{10}
$$

For the ferromagnetic interaction, we have J, K and $L>0$. In the first equation of (10), only ($J^*=0$, K^*, L^*) is a possible solution for $q>1$. Substituting it to the second equation of (10), we obtain the solution ($J^*=0$, $K^*=0$, L^*). Similarly, substituting the latter in the third equation of (10), we finally reach the unique solution ($J^*=K^*=L^*=0$) which corresponds to a disordered phase at high temperature.

In summary, we have performed a decimation renormalisation for the Potts model on the Koch curve including all the nearest-neighbour interactions, through introducing three kinds of coupling parameters J, K and L. The recursion equations are produced and their fixed points are found. We have noted that both fixed points $J^*=K^*=L^*=0$ and $J^*=K^*=L^*=\infty$ are trivial, which shows that no phase transition occurs at finite temperature.

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